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Liouville–Green Approximations via the Riccati Transformation

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The Riccati transformation is used to obtain Liouville–Green approximations, along with error bounds possessing a particularly simple and explicit form, for the equation $u'' = f(t)\,u$. The resulting error bounds involve the function f and its first derivative, whereas the earlier bounds of Olver [1961, 1974] and Taylor [1978, 1982] involve also the second derivative of f. The theorems given here resemble corresponding results of Olver [1974, Chapter 6], and indeed the present work owes a great debt to Olver. The Riccati transformation used here is based on work of Y. Sibuya, K. W. Chang, and W. A. Harris, and can be used in a more general study of linear ordinary differential equations and systems of such equations (cf. Smith [1984, 1985]).

1. INTRODUCTION

Consider the scalar second order equation

$$\frac{d^2u}{dt^2} = f(t)\,u, \tag{1.1}$$

where $f = f(t)$ is a given piecewise differentiable real or complex function which for the moment can be thought of as being nonzero on a given bounded or unbounded real interval (a, b). Specifically, following Olver [9, 10], the function f is assumed to be one of the following two types, either

$$f(t) = p(t)^2 + q(t) \qquad \text{for} \quad t \in (a, b), \tag{1.2}$$

or

$$f(t) = -p(t)^2 + q(t) \qquad \text{for} \quad t \in (a, b), \tag{1.3}$$

where in each case p is a given real-valued piecewise differentiable function which is positive,

$$p(t) > 0 \qquad \text{for} \quad t \in (a, b), \tag{1.4}$$

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and q is a given real or complex piecewise continuous function that is small compared to p in a sense to be made precise below.

Hence Eq. (1.1) becomes either

$$\frac{d^2u}{dt^2} - p(t)^2 u = q(t) u \qquad \text{for} \quad t \in (a, b) \tag{1.5}$$

or

$$\frac{d^2u}{dt^2} + p(t)^2 u = q(t) u \qquad \text{for} \quad t \in (a, b), \tag{1.6}$$

and in each case it is convenient to consider also the following corresponding first order system, either

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p(t)^2 + q(t) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{for} \quad t \in (a, b) \tag{1.7}$$

or

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p(t)^2 + q(t) & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{for} \quad t \in (a, b). \tag{1.8}$$

Equation (1.5) or (1.7) is referred to as the *nonoscillatory* case while (1.6) or (1.8) is referred to as the *oscillatory* case, even though solutions need not oscillate in the latter case, as illustrated by the example (see p. 190 of Olver [10]),

$$\frac{d^2u}{dt^2} + \frac{\alpha \cdot (1-\alpha)}{t^2} u = 0 \qquad \text{for} \quad t > 0. \tag{1.9}$$

This latter equation is of the oscillatory type (1.6) in the case $0 < \alpha < 1$, with

$$p(t) := \frac{1}{t}\sqrt{\alpha \cdot (1-\alpha)} \quad \text{and} \quad q(t) := 0 \quad \text{for } t > 0, 0 < \alpha < 1. \tag{1.10}$$

However, (1.9) has solutions which do not oscillate, given as

$$u_1(t) = t^{\alpha}, \qquad u_2(t) = \begin{cases} t^{1/2} \log t & \text{for } \alpha = \frac{1}{2} \\ t^{1-\alpha} & \text{otherwise,} \end{cases} \tag{1.11}$$

for $t > 0$, $0 < \alpha < 1$.

The Riccati transformation is used here to provide fundamental solutions in a particularly convenient form for the systems (1.7) and (1.8), yielding thereby a new derivation for the result that any solution of (1.5) or (1.6) can be represented in the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) \tag{1.12}$$

for suitable solutions u_1 and u_2 satisfying, respectively,

$$\begin{aligned} u_1(t) &= \frac{1}{\sqrt{p(t)}} e^{-\int p}[1 + E_1(t)], \quad u_1'(t) = -\sqrt{p(t)}\, e^{-\int p}[1 + F_1(t)], \\ u_2(t) &= \frac{1}{\sqrt{p(t)}} e^{+\int p}[1 + E_2(t)], \quad u_2'(t) = +\sqrt{p(t)}\, e^{+\int p}[1 + F_2(t)], \end{aligned} \tag{1.13}$$

for the nonoscillatory case (1.5), and

$$\begin{aligned} u_1(t) &= \frac{1}{\sqrt{p(t)}} e^{-i\int p}[1 + E_1(t)], \quad u_1'(t) = -i\sqrt{p(t)}\, e^{-i\int p}[1 + F_1(t)], \\ u_2(t) &= \frac{1}{\sqrt{p(t)}} e^{+i\int p}[1 + E_2(t)], \quad u_2'(t) = +i\sqrt{p(t)}\, e^{+i\int p}[1 + F_2(t)], \end{aligned} \tag{1.14}$$

for the oscillatory case (1.6), where the error functions E_j, F_j $(j = 1, 2)$ can be represented in terms of p, p', and q, yielding explicit error bounds subject to appropriate conditions on the data. In fact the method yields information on u_1 and u_2 that is considerably more precise than indicated by (1.13) and (1.14).

We obtain (improved) results of the general type as (1.13) and (1.14) on any suitable subinterval (c, d),

$$(c, d) \subset (a, b), \tag{1.15}$$

and we give a sufficient condition that permits one to take $c = a$ and $d = b$, in which latter case the subinterval coincides with the entire interval (a, b). The expressions (c, d) and (a, b) are used here to denote real intervals that may be bounded or unbounded.

Suitable fundamental solutions for the systems (1.7) and (1.8) are obtained in Section 2 using the Riccati transformation, and then these fundamental solutions are used in Sections 3 and 4 to provide Liouville–Green approximations for (1.5) and (1.6).

The present approach complements earlier results of Olver [9, 10] and Taylor [15, 16] where error bounds were obtained by a different method subject to slightly stronger smoothness requirements on the data. The present weakening of the smoothness requirements plays an important role in certain applications in singular perturbation theory (cf. Exercises 9.2.8, 9.3.7, 9.3.10, 10.2.3, and 10.2.7, and Section 10.3, in Smith [14]). Analytic examples only are given in Sections 3 and 4.

There is also a body of results following Perron [11], including results of Levinson [8], Hartman and Wintner [7], Bellman [2], Atkinson [1], Coppel [3, 4], Harris and Lutz [5, 6], and others cited in these references, and these results can be applied to systems such as (2.2) so as to yield various Liouville–Green approximations, again subject to suitable regularity requirements on the data. The present approach based on the Riccati transformation complements these latter results by providing a particularly simple and explicit form for the error bounds and, in some cases, by weakening the smoothness requirements on the data.

2. The Fundamental Solutions

First, consider the nonoscillatory system (1.7). Suitable transformations are sought so as to diagonalize this system. The system is first transformed with the eigenvector transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -p(t) & p(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{1}{p(t)} \\ 1 & +\dfrac{1}{p(t)} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.1}$$

yielding the transformed system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \left[\begin{pmatrix} -p(t) & 0 \\ 0 & p(t) \end{pmatrix} + \frac{p'(t)}{2p(t)}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} + \frac{q(t)}{2p(t)}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.2}$$

This latter system is already in diagonal form if and only if the coefficient function q vanishes and the function p is constant, with $q \equiv 0$ and $p' \equiv 0$. Otherwise (2.2) is a coupled system for x and y, in which case we apply a Riccati transformation so as to uncouple this last system.

The following Riccati transformation (cf. Smith [13, 14])

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -T & 1 \end{pmatrix}\begin{pmatrix} 1 & -S \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -S(t) \\ -T(t) & 1 + T(t)\,S(t) \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \end{aligned} \tag{2.3a}$$

with

$$\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ T & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 1 + S(t)\,T(t) & S(t) \\ T(t) & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.3b}$$

is applied to (2.2), yielding the uncoupled system

$$\begin{aligned} \frac{d\hat{x}}{dt} &= -\left[p + \frac{q+p'}{2p} + \left(\frac{p'-q}{2p} \right) T(t) \right] \hat{x}, \\ \frac{d\hat{y}}{dt} &= +\left[p + \frac{q-p'}{2p} + \left(\frac{p'-q}{2p} \right) T(t) \right] \hat{y}, \end{aligned} \tag{2.4}$$

provided that the functions T and S satisfy the equations

$$\frac{dT}{dt} - \left[2p(t) + \frac{q(t)}{p(t)} \right] T + \left[\frac{p'(t)+q(t)}{2p(t)} \right] = \left[\frac{p'(t)-q(t)}{2p(t)} \right] T^2 \tag{2.5}$$

and

$$\frac{dS}{dt} + \left[2p(t) + \frac{q(t)}{p(t)} + \left(\frac{p'(t)-q(t)}{p(t)} \right) T(t) \right] S + \left[\frac{p'(t)-q(t)}{2p(t)} \right] = 0. \tag{2.6}$$

First, we obtain a suitable solution for the nonlinear equation (2.5) for $T = T(t)$, and then we use this solution for T in (2.6) and solve the resulting linear equation for S.

To obtain a suitable solution T for Eq. (2.5) on a subinterval (c, d) as in (1.15), we impose the terminal condition $T(d) = 0$ and replace the resulting terminal value problem with the following equivalent integral equation

$$T(t) = T_0(t) - \int_t^d e^{-\int_t^s (2p + (q/p))} \left[\frac{p'(s)-q(s)}{2p(s)} \right] T(s)^2 \, ds, \tag{2.7}$$

where T_0 is defined as

$$T_0(t) \equiv T_0(t, d) := \int_t^d e^{-\int_t^s (2p + (q/p))} \left[\frac{p'(s)+q(s)}{2p(s)} \right] ds. \tag{2.8}$$

In the study of (2.7), (2.8) it is useful to introduce the nonnegative-valued function $\kappa(t) = \kappa(t, d; p, q)$ defined as

$$\kappa(t) \equiv \kappa(t, d; p, q) := \int_t^d e^{-\int_t^s [2p + (\mathrm{Re}\, q/p)]} \left(\frac{|p'(s)| + |q(s)|}{p(s)} \right) ds \tag{2.9}$$

for $t \in (c, d)$, whenever the right side of (2.9) is well defined. Note that this quantity κ is certainly well defined for all positive-valued p for which there hold

$$\frac{p'}{p} \in L_1(c, d) \quad \text{and} \quad \frac{q}{p} \in L_1(c, d), \tag{2.10}$$

although (2.9) is also well defined in many situations for which (2.10) fails to hold. For example, the functions $p(t) := t$, $q(t) := 1$ for $t \in (0, \infty)$ are easily seen to satisfy $\kappa(t, \infty; p, q) \leqslant 2 \cdot t^{-2}$ for $0 < t < \infty$, although (2.10) is not satisfied here with $d = \infty$.

We now show that the integral equation (2.7)–(2.8), and hence also the Riccati differential equation (2.5), has a suitable solution.

LEMMA 2.1 (Nonoscillatory case). *Let the real or complex function q be piecewise continuous and let the real positive-valued function p be piecewise differentiable on the (bounded or unbounded) interval (c, d), and let there hold*

$$\sup_{c<t<d} \kappa(t) < 1 \tag{2.11}$$

with κ defined by (2.9). Then the integral equation (2.7), (2.8), and hence also the differential equation (2.5), has a continuous, piecewise differentiable (real or complex) solution on (c, d) satisfying the estimate

$$|T(t)| \leqslant \kappa(t, d; p, q) \qquad \text{for} \quad t \in (c, d). \tag{2.12}$$

If q is real, then so is T.

Proof. Let the operator $\mathfrak{R}$ be defined as

$$\mathfrak{R}T(t) := T_0(t) - \int_t^d e^{-\int_t^s (2p + (q/p))} \left[\frac{p'(s) - q(s)}{2p(s)}\right] T(s)^2\, ds \quad \text{for } t \in (c, d) \tag{2.13}$$

on the Banach space $\mathfrak{B}$ consisting of all (real or complex) functions T of class $C^0(c, d)$, with norm

$$\|T\| := \sup_{c<t<d} |T(t)|, \tag{2.14}$$

and where T_0 is given by (2.8). Then Eqs. (2.7), (2.8) on (c, d) can be written as the fixed point equation

$$T = \mathfrak{R}T. \tag{2.15}$$

The function T_0 of (2.8) is continuous, piecewise differentiable and bounded, with $|T_0(t)| \leqslant \frac{1}{2}\kappa(t)$, so that (2.11) implies

$$\|T_0\| < \tfrac{1}{2}. \tag{2.16}$$

Let $\mathfrak{B}_\rho$ be the ball centered at the origin of radius ρ in the Banach space $\mathfrak{B}$,

$$\mathfrak{B}_\rho := \{T \in \mathfrak{B} \mid \|T\| \leqslant \rho\}.$$

A routine calculation using (2.16) and (2.11) shows now that the operator $\mathfrak{N}$ is a contracting operator that maps $\mathfrak{B}_\rho$ into itself if the radius ρ is given as $\rho := 2\|T_0\| < 1$.

It follows then directly from the Banach–Picard fixed point theorem that Eq. (2.15) has a unique solution on the stated ball, and the solution satisfies the estimate

$$\sup_{c<t<d} |T(t)| \leqslant 1. \tag{2.17}$$

The bound (2.17) can be used back in the right side of the integral equation (2.7), (2.8), and one finds directly the estimate (2.12). Finally, the resulting solution T is clearly continuous and piecewise differentiable. This completes the proof of the lemma. ∎

Given the solution T of the Riccati equation (2.5) as provided by Lemma 2.1, we now define the following corresponding solution S of (2.6) as

$$S(t) := -\int_c^t e^{-\int_s^t [2p + (q/p) + ((p'-q)/p)T]} \left(\frac{p'(s) - q(s)}{2p(s)}\right) ds \qquad \text{for} \quad t \in (c, d), \tag{2.18}$$

where we have imposed the initial condition $S(c) = 0$.

For the uncoupled system (2.4), we now use the fundamental solution $\hat{Z} = \hat{Z}(t)$ given, up to an inessential multiplicative constant, as (note the result $\exp[-\frac{1}{2}\int_c^t (p'/p)] = [p(c)/p(t)]^{1/2}$),

$$\hat{Z}(t) = \frac{2}{\sqrt{p(t)}} \begin{pmatrix} e^{-\int_c^t [p + (q/2p) + ((p'-q)/2p)T]} & 0 \\ 0 & e^{+\int_c^t [p + (q/2p) + (p'-q)/2p)T]} \end{pmatrix} \tag{2.19}$$

for $t \in (c, d)$, where the lower limit of integration c in the integrals here can be replaced in either one or both integrals by any fixed $t_1 \in (c, d)$. The transformations (2.1) and (2.3) lead directly now to the following fundamental solution W for (1.7), given as

$$\begin{aligned} W(t) &= \text{Fundamental solution for the nonoscillatory system (1.7)} \\ &= \frac{1}{2} \begin{pmatrix} 1 - T(t) & 1 + [T(t) - 1]\, S(t) \\ -p(t)[1 + T(t)] & p(t)[1 + (T(t) + 1)\, S(t)] \end{pmatrix} \hat{Z}(t), \end{aligned} \tag{2.20}$$

where $\hat{Z}$ is given by (2.19), and where T is the function given by Lemma 2.1 and S is given by (2.18).

Now, we turn to a consideration of the oscillatory system (1.8), which is obtained from the nonoscillatory system (1.7) by replacing p with ip, $i \equiv \sqrt{-1}$. One sees that the above discussion given for (1.7) applies also to (1.8) if one replaces p everywhere with ip in (2.1)–(2.6). Hence, instead of the previous integral equation (2.7), (2.8) for the function T, we now use the following equation in the oscillatory case,

$$T(t) = T_0(t) + \int_{t_0}^{t} e^{i\int_s^t (2p - (q/p))} \left[\frac{p'(s) + iq(s)}{2p(s)}\right] T(s)^2 \, ds, \tag{2.21}$$

where T_0 is defined now as

$$T_0(t) \equiv T_0(t, t_0) := -\int_{t_0}^{t} e^{i\int_s^t (2p - (q/p))} \left[\frac{p'(s) - iq(s)}{2p(s)}\right] ds, \tag{2.22}$$

and where t_0 is an arbitrary, fixed number in the closure of a given interval (c, d). (The possibilities $t_0 = -\infty$ or $t_0 = +\infty$ are included.) The function T is to be obtained now as the solution of (2.21), (2.22), and then S is given as

$$S(t) := -\int_{t_0}^{t} e^{-i\int_s^t [2p - (q/p) + ((q - ip')/p)T]} \left(\frac{p'(s) + iq(s)}{2p(s)}\right) ds \tag{2.23}$$

for $t \in (c, d)$, where the initial condition $S(t_0) = 0$ has been imposed.

In the study of (2.21), (2.22) it is useful to introduce the nonnegative-valued function $\mu(t) = \mu(t, t_0; p, q)$ defined as

$$\mu(t) \equiv \mu(t, t_0; p, q) := \left| \int_{t_0}^{t} e^{\int_s^t (\operatorname{Im} q/p)} \left(\frac{|p'(s)| + |q(s)|}{p(s)}\right) ds \right| \tag{2.24}$$

for $t \in (c, d)$, whenever the right side is well defined.

The following lemma gives sufficient conditions for the existence of a suitable solution T.

LEMMA 2.2 (Oscillatory case). *Let the real or complex function q be piecewise continuous and let the real positive-valued function p be piecewise differentiable on the (bounded or unbounded) interval (c, d), and let there hold*

$$\sup_{c < t < d} \mu(t) < 1 \tag{2.25}$$

for a fixed (finite or infinite) t_0 in the closure of (c, d), where μ is defined by

(2.24). *Then the integral equation* (2.21), (2.22) *has a continuous, piecewise differentiable solution on* (c, d) *satisfying the estimate*

$$|T(t)| \leqslant \mu(t, t_0; p, q) \qquad \text{for} \quad t \in (c, d). \tag{2.26}$$

Proof. The obvious modification of the proof of Lemma 2.1 suffices. ∎

In this case one finds the following fundamental solution W for (1.8) (replace p with ip in (2.20))

$$\begin{aligned} W(t) &= \text{Fundamental solution for the oscillatory system (1.8)} \\ &= \frac{1}{2}\begin{pmatrix} 1 - T(t) & 1 + [T(t) - 1]\, S(t) \\ -ip(t)[1 + T(t)] & ip(t)[1 + (T(t) + 1)\, S(t)] \end{pmatrix} \hat{Z}(t) \end{aligned} \tag{2.27}$$

for $t \in (c, d)$, where $\hat{Z}$ is given here as

$$\hat{Z}(t) = \frac{2}{\sqrt{p(t)}} \begin{pmatrix} e^{-i\int_{t_0}^{t}[p - (q/2p) + ((q - ip')/2p)T]} & 0 \\ 0 & e^{+i\int_{t_0}^{t}[p - (q/2p) + ((q - ip')/2p))T]} \end{pmatrix}, \tag{2.28}$$

and where T is the function given by Lemma 2.2 and S is given by (2.23).

3. Liouville–Green Approximations: Nonoscillatory Case

The data are assumed here to satisfy the following conditions:

$$\int_t^d e^{-\int_t^s [2p + (\operatorname{Re} q/p)]} \left(\frac{|p'(s)| + |q(s)|}{p(s)} \right) ds = o(1)$$

and (3.1)

$$\int_t^d \left(\frac{|p'(s)| + |q(s)|}{p(s)} \right) \kappa(s)\, ds = o(1), \quad \text{both as } t \to d^-,$$

and

$$\int_c^t e^{-\int_s^t [2p + (\operatorname{Re} q/p)]} \left(\frac{|p'(s)| + |q(s)|}{p(s)} \right) ds = o(1)$$

and (3.2)

$$\int_c^t \left(\frac{|p'(s)| + |q(s)|}{p(s)}\right) \kappa(s)\, ds = o(1) \quad \text{both as } t \to c^+,$$

where κ is defined by (2.9).

THEOREM 3.1. *Let the conditions of Lemma* 2.1 *hold, and assume that both* (3.1) *and* (3.2) *hold. Then every solution of the equation*

$$\frac{d^2u}{dt^2} - p(t)^2\, u = q(t)\, u \qquad \text{for} \quad t \in (c, d) \tag{3.3}$$

can be represented in the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) \tag{3.4}$$

for suitable solutions u_1 and u_2 satisfying, respectively,

$$\begin{aligned} u_1(t) &= (1/\sqrt{p(t)})\, e^{-\int (p + (q/2p))}[1 + E_1(t, d)] \\ u_1'(t) &= -\sqrt{p(t)}\, e^{-\int (p + (q/2p))}[1 + F_1(t, d)] \end{aligned} \tag{3.5}$$

with

$$\begin{aligned} E_1(t, d) &= [e^{\int_t^d ((p' - q)/2p)T} - 1] - T(t)\, e^{\int_t^d ((p' - q)/2p)T}, \\ F_1(t, d) &= [e^{\int_t^d ((p' - q)/2p)T} - 1] + T(t)\, e^{\int_t^d ((p' - q)/2p)T}, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} u_2(t) &= (1/\sqrt{p(t)})\, e^{+\int (p + (q/2p))}[1 + E_2(t, c)], \\ u_2'(t) &= \sqrt{p(t)}\, e^{+\int (p + (q/2p))}[1 + F_2(t, c)], \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} E_2(t, c) &= [e^{\int_c^t ((p' - q)/2p)T} - 1] + [T(t) - 1]\, S(t)\, e^{\int_c^t ((p' - q)/2p)T}, \\ F_2(t, c) &= [e^{\int_c^t ((p' - q)/2p)T} - 1] + [T(t) + 1]\, S(t)\, e^{\int_c^t ((p' - q)/2p)T}, \end{aligned} \tag{3.8}$$

where T satisfies (2.12) *and S is given by* (2.18). *The error terms E_j, F_j ($j = 1, 2$) satisfy $E_1(t, d)$, $F_1(t, d) \to 0$ as $t \to d^-$, and $E_2(t, c)$, $F_2(t, c) \to 0$ as $t \to c^+$. Moreover, E_1 and F_1 are uniformly small for fixed c if d is taken sufficiently close to c, while E_2 and F_2 are uniformly small for fixed d if c is* *taken sufficiently close to d. For example, for given (finite or infinite) d and for any given* $\varepsilon > 0$, *there is a fixed* $c < d$ *such that there holds*

$$|E_2(t)|,\ |F_2(t)| \leqslant \varepsilon \qquad \text{for} \quad t \in [c, d). \tag{3.9}$$

Proof. The first column vector of the fundmanental solution (2.20) leads directly to the solution (3.5), (3.6) up to an inessential multiplicative constant. Specifically, one multiplies the first column of (2.20) by the constant

$$\exp\left[-\int_c^d \left(\frac{p'-q}{2p}\right) T\right],$$

where the stated assumptions with (2.12) imply the result $((p'-q)/2p)\, T \in L_1(c, d)$. Similarly one finds the solution (3.7), (3.8) from the second column of (2.20).

The following estimates are obtained now directly from (2.12), (2.18), (3.1), (3.2), (3.6), (3.8) and the inequality $|e^x - 1| \leqslant |x| \cdot \exp(|x|)$,

$$|E_1(t, d)|,\ |F_1(t, d)| \leqslant \left[\kappa(t) + \int_t^d \left(\frac{|p'| + |q|}{2p}\right) \kappa\right] e^{\int_t^d((|p'| + |q|)/2p)\kappa} = o(1) \tag{3.10}$$

as $t \to d^-$, and

$$|E_2(t, c)|,\ |F_2(t, c)| \leqslant \left[(1 + \kappa(t))\,|S(t)| + \int_c^t \left(\frac{|p'| + |q|}{2p}\right) \kappa\right] e^{\int_c^t((|p'| + |q|)/2p)\kappa} = o(1) \tag{3.11}$$

as $t \to c^+$, where this last $o(1)$ result follows in part from the inequality (cf. (2.18)),

$$|S(t)| \leqslant \frac{1}{2}\, e^{\int_c^t((|p'| + |q|)/2p)\kappa} \int_c^t e^{-\int_s^t[2p + (\mathrm{Re}\, q/p)]} \left(\frac{|p'(s)| + |q(s)|}{p(s)}\right) ds \tag{3.12}$$

for $t \in (c, d)$. The remaining assertions of the theorem follow directly from the stated assumptions along with (3.6) and (3.8). This completes the proof of the theorem. ∎

EXAMPLE 3.1. The differential equation for the *Weber parabolic cylinder function* is (cf. Olver [10])

$$\frac{d^2u}{dt^2} = \left(\frac{1}{4}t^2 + \lambda\right) u, \tag{3.13}$$

where λ is a fixed parameter. Equation (3.13) is of the form (1.5) with

$$p(t) := \tfrac{1}{2}t \qquad \text{and} \qquad q(t) := \lambda. \tag{3.14}$$

We consider solutions of (3.13) for $t \in [c, \infty)$ for suitable $c > 0$. Note that the integrability conditions of (2.10) do *not* hold (with $c > 0$, $d = \infty$), but one finds easily with a direct estimation from (2.9) the inequality

$$0 \leqslant \kappa(t) \leqslant \frac{1 + 2|\lambda|}{t^2} \qquad \text{for} \quad t > 0, \lambda \geqslant -1. \tag{3.15}$$

Hence the condition (2.11) is satisfied with $d = \infty$, for any fixed c with

$$c > [1 + 2|\lambda|]^{1/2}. \tag{3.16}$$

We assume that (3.16) holds, so that we also have the results of Lemma 2.1. Moreover one finds easily with (2.12) the inequality

$$|T(t)| \leqslant \frac{1 + 2|\lambda|}{t^2} \qquad \text{for} \quad t \in [c, \infty), \tag{3.17}$$

and one sees also that (3.1) and (3.2) hold, so that Theorem 3.1 applies.

In particular one finds (up to an inessential multiplicative constant) the principal solution $u_1 = u_1(\lambda, t)$ that is recessive at infinity, satisfying (cf. (3.5)),

$$\begin{aligned} u_1(\lambda, t) &= t^{-(\lambda + (1/2))} e^{-t^2/4} [1 + E_1(t)], \\ \frac{du_1(\lambda, t)}{dt} &= -\frac{1}{2} t^{-(\lambda - (1/2))} e^{-t^2/4} [1 + F_1(t)], \end{aligned} \tag{3.18}$$

where $E_1(t) \equiv E_1(\lambda, t)$ and $F_1(t) \equiv F_1(\lambda, t)$ satisfy (cf. (3.6)),

$$|E_1(t)|, |F_1(t)| \leqslant \frac{(1 + 2|\lambda|)(5 + 2|\lambda|)}{4t^2} \exp\left(\frac{1 + 2|\lambda|}{2t}\right)^2 \tag{3.19}$$

for $t \in [c, \infty)$ and $\lambda \geqslant -1$, where (3.16) is assumed to hold. In particular one has, for any such fixed λ, the results $E_1(t)$, $F_1(t) = O(t^{-2})$ as $t \to \infty$.

The results of (3.19) and similar results obtained below can in some cases be improved. For example, (3.19) gives for $|E_1(t)|$ and $|F_1(t)|$ a bound that is asymptotically (as $t \to \infty$)

$$(|\lambda| + \tfrac{1}{2}) \cdot (|\lambda| + \tfrac{5}{2}) \cdot t^{-2}.$$

On the other hand, in this case one can return to the proof of Lemma 2.1 and tighten the analysis so as to replace this latter asymptotic result with the better result

$$|\lambda + \tfrac{1}{2}| \cdot |\lambda + \tfrac{3}{2}| \cdot t^{-2}.$$

By way of comparison, in this case the method of Olver [9, 10] yields a bound for $|E_1(t)|$ which has the asymptotic form

$$\tfrac{1}{2}\cdot|\lambda+\tfrac{1}{2}|\cdot|\lambda+\tfrac{3}{2}|\cdot t^{-2},$$

which is sharp in the sense that it agrees with the true asymptotic form of the absolute value of the error.

Continuing with Example (3.13), one also finds (again up to an inessential multiplicative constant) the following solution $u_2 = u_2(\lambda, t)$ that increases as $t \to \infty$ (cf. (3.7)),

$$\begin{aligned} u_2(\lambda, t) &= t^{+(\lambda-(1/2))} e^{t^2/4}[1 + E_2(t)], \\ \frac{du_2}{dt}(\lambda, t) &= \tfrac{1}{2} t^{+(\lambda+(1/2))} e^{t^2/4}[1 + F_2(t)], \end{aligned} \tag{3.20}$$

where $E_2(t) \equiv E_2(\lambda, t)$ and $F_2(t) \equiv F_2(\lambda, t)$ satisfy (cf. (3.8) and (3.12))

$$\begin{aligned} &|E_2(t)|, |F_2(t)| \\ &\quad \leqslant \frac{1+2|\lambda|}{c^2}\left[\frac{1+2|\lambda|}{4} + \exp\left(\frac{1+2|\lambda|}{2t}\right)^2\right]\exp\left(\frac{1+2|\lambda|}{2t}\right)^2 \end{aligned} \tag{3.21}$$

for $t \in [c, \infty)$ and $\lambda \geqslant 1$. In particular, for any large $c (c \to \infty)$, one has the result $|E_2(t)|$, $|F_2(t)| = O(c^{-2})$ uniformly for $t \geqslant c$, which provides an explicit version of the result discussed above in connection with (3.9).

EXAMPLE 3.2. Consider the *Airy equation*

$$\frac{d^2u}{dt^2} = tu, \tag{3.22}$$

which is nonoscillatory for $t > 0$. In the notation of (1.5) and (1.7) one has

$$p(t) := \sqrt{t} \qquad \text{and} \qquad q(t) := 0 \quad \text{for } t > 0. \tag{3.23}$$

One finds easily with a direct estimation from (2.9) and (3.23) the inequality

$$0 \leqslant \kappa(t) \leqslant \tfrac{1}{4} t^{-(3/2)} \qquad \text{for} \quad t > 0, \tag{3.24}$$

so that (2.11) holds with $d = \infty$ and any fixed c satisfying

$$c > (16)^{-1/3}. \tag{3.25}$$

We assume that (3.25) holds, so that the results of Lemma 2.1 hold on the interval $[c, \infty)$. One also finds directly that the conditions (3.1) and (3.2) hold, so that Theorem 3.1 applies.

In this way we find (up to multiplicative constants) solutions u_1 and u_2 satisfying (cf. (3.5)–(3.8)),

$$\begin{aligned} u_1(t) &= t^{-1/4}e^{-(2/3)t^{3/2}}[1+E_1(t)], \\ \frac{du_1(t)}{dt} &= -t^{1/4}e^{-(2/3)t^{3/2}}[1+F_1(t)] \end{aligned} \tag{3.26}$$

for $t \geqslant (16)^{-1/3}$, and

$$\begin{aligned} u_2(t) &= t^{-1/4}e^{(2/3)t^{3/2}}[1+E_2(t)], \\ \frac{du_2(t)}{dt} &= t^{1/4}e^{(2/3)t^{3/2}}[1+F_2(t)] \end{aligned} \tag{3.27}$$

for $t \geqslant (16)^{-1/3}$, where $E_1(t)$ and $F_1(t)$ satisfy

$$|E_1(t)|,\ |F_1(t)| \leqslant \frac{7}{24}\, t^{-3/2} \exp\left(\frac{1}{24}\, t^{-3/2}\right) \tag{3.28}$$

for $t \geqslant (16)^{-1/3}$, while $E_2(t)$ and $F_2(t)$ satisfy

$$|E_2(t)|,\ |F_2(t)| \leqslant \frac{1}{2}\, c^{-3/2}\left[\frac{1}{12} + \exp\left(\frac{1}{24}\, c^{-3/2}\right)\right] \exp\left(\frac{1}{24}\, c^{-3/2}\right) \tag{3.29}$$

uniformly for $t \geqslant c \geqslant (16)^{-1/3}$, where one can replace the strict inequality of (3.25) with the weak inequality $c \geqslant (16)^{-1/3}$. In particular there holds $E_1(t)$, $F_1(t) = O(t^{-3/2})$ as $t \to \infty$, and $E_2(t)$, $F_2(t) = O(c^{-3/2})$ for $t \geqslant c$ (fixed, large positive c).

The present derivation yields the solutions u_1 and u_2 of (3.26) and (3.27) for

$$t \in [c_1, \infty) \qquad \text{with} \quad c_1 = (16)^{-1/3} \doteq 0.39685. \tag{3.30}$$

The derivation can be repeated on a contiguous interval $[c_2, c_1]$ for a suitable positive $c_2 < c_1$, and indeed this approach can be used to continue the solutions u_1 and u_2 everywhere on $t > 0$.

4. Liouville–Green Approximations: Oscillatory Case

The data are assumed here to satisfy the following conditions:

$$\begin{aligned} &\mu(t) \equiv \mu(t, t_0; p, q) = o(1), \\ &\int_{t_0}^{t} \left(\frac{|p'(s)| + |q(s)|}{p(s)}\right) \mu(s)\, ds = o(1) \ \text{ both as } t \to t_0, \end{aligned} \tag{4.1}$$

where μ is defined by (2.24).

THEOREM 4.1. *Let the conditions of Lemma 2.2 hold, and assume also that the conditions of* (4.1) *hold. Then every solution of the equation*

$$\frac{d^2u}{dt^2} + p(t)^2 u = q(t) u \qquad \text{for} \quad t \in (c, d) \tag{4.2}$$

can be represented in the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) \tag{4.3}$$

for suitable solutions u_1 and u_2 satisfying respectively

$$\begin{aligned} u_1(t) &= (1/\sqrt{p(t)})\, e^{-i\int(p-(q/2p))}[1 + E_1(t, t_0)] \\ u_1'(t) &= -i\sqrt{p(t)}\, e^{-i\int(p-(q/2p))}[1 + F_1(t, t_0)] \end{aligned} \tag{4.4}$$

with

$$\begin{aligned} E_1(t, t_0) &= [e^{-\int_{t_0}^{t}((p'+iq)/2p)T} - 1] - T(t)\, e^{-\int_{t_0}^{t}((p'+iq)/2p)T}, \\ F_1(t, t_0) &= [e^{-\int_{t_0}^{t}((p'+iq)/2p)T} - 1] + T(t)\, e^{-\int_{t_0}^{t}((p'+iq)/2p)T}, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} u_2(t) &= (1/\sqrt{p(t)})\, e^{+i\int(p-(q/2p))}[1 + E_2(t, t_0)], \\ u_2'(t) &= +i\sqrt{p(t)}\, e^{+i\int(p-(q/2p))}[1 + F_2(t, t_0)], \end{aligned} \tag{4.6}$$

with

$$\begin{aligned} E_2(t, t_0) &= [e^{\int_{t_0}^{t}((p'+iq)/2p)T} - 1] + [T(t) - 1]\, S(t)\, e^{\int_{t_0}^{t}((p'+iq)/2p)T}, \\ F_2(t, t_0) &= [e^{\int_{t_0}^{t}((p'+iq)/2p)T} - 1] + [T(t) + 1]\, S(t)\, e^{\int_{t_0}^{t}((p'+iq)/2p)T}, \end{aligned} \tag{4.7}$$

for any fixed (finite or infinite) t_0 in the closure of (c, d), where T satisfies (2.26) *and S is given by* (2.23). *The error terms E_j, F_j ($j = 1, 2$) satisfy*

$$E_j(t, t_0),\ F_j(t, t_0) \to 0 \qquad \text{as} \quad t \to t_0, \tag{4.8}$$

and in fact (4.5) *and* (4.7) *provide explicit error bounds.*

Proof. The obvious modification of the proof of Theorem 3.1 suffices, using Lemma 2.2 instead of Lemma 2.1, with the previous fundamental solution (2.20) replaced now by (2.27). ▮

EXAMPLE 4.1. Consider the oscillatory equation (1.6) with *constant* $p(t)$, say $p(t) \equiv 1$ (all t), so that the equation becomes

$$\frac{d^2u}{dt^2} + u = q(t) u \qquad \text{for} \quad t \in (a, \infty) \tag{4.9}$$

for any fixed $a \in \mathbb{R}$ and for a given piecewise continuous real-valued function q which is assumed to be integrable,

$$q \in L_1(a, \infty). \tag{4.10}$$

In this case it follows directly from (2.24) and (4.10) that we can take $t_0 = \infty$, and the resulting function μ is given as

$$\mu(t) = \int_t^\infty |q(s)|\, ds \qquad \text{for} \quad t \in (a, \infty). \tag{4.11}$$

The integrability condition (4.10) implies the result

$$\int_c^\infty |q| < 1 \qquad \text{for all large enough } c \in (a, \infty), \tag{4.12}$$

so that (2.25) holds for any such c, with $t_0 = d = \infty$ and μ given by (4.11). Hence Lemma 2.2 applies, and (2.26) and (2.23) lead directly to the bounds

$$\begin{aligned} |T(t)| &\leqslant \int_t^\infty |q| < 1, \\ |S(t)| &\leqslant \frac{1}{2}\int_t^\infty e^{\int_t^s |q|} |q(s)|\, ds = \frac{1}{2}\left[e^{\int_t^\infty |q|} - 1\right] \leqslant \frac{e-1}{2} \end{aligned} \tag{4.13}$$

for $t \geqslant c$, where c is any fixed number satisfying (4.12).

Theorem 4.1 now applies, and one has the solutions u_1 and u_2 of (4.4) and (4.6), given as

$$\begin{aligned} u_1(t) &= e^{-it} e^{i\int q/2}[1 + E_1(t)], \\ u_2(t) &= e^{+it} e^{-i\int q/2}[1 + E_2(t)] \qquad \text{for} \quad t \geqslant c, \end{aligned} \tag{4.14}$$

where the functions E_1 and E_2 are given by (4.5) and (4.7). It follows from (4.12) and (4.13) that $E_1(t)$ and $E_2(t)$ satisfy explicit bounds which yield in particular

$$E_1(t), E_2(t) = O\left(\int_t^\infty |q|\right) = o(1) \qquad \text{as} \quad t \to \infty. \tag{4.15}$$

Hence one has a direct verification of the well-known result (cf. p. 112 of Bellman [2]) that *all solutions of* (4.9) *are bounded as* $t \to \infty$ *if* q *satisfies the integrability condition* (4.10). In this case the present approach based on the Riccati transformation yields the more precise result that (4.9) has linearly independent solutions u_1 and u_2 satisfying

$$
\begin{aligned}
u_1(t) &= e^{-i[t+\int_t^\infty (q/2)]}\left[1+O\left(\int_t^\infty |q|\right)\right],\\
u_2(t) &= e^{+i[t+\int_t^\infty (q/2)]}\left[1+O\left(\int_t^\infty |q|\right)\right]
\end{aligned} \tag{4.16}
$$

as $t \to \infty$.

The condition (4.10) is sufficient to guarantee the boundedness of solutions. In fact (4.10) comes close to being sharp, as indicated by the following example of Perron [12] (cf. Bellman [2]), with

$$q(t) := -\left[\frac{2\sin 2t}{t} + \left(\frac{\sin 2t}{2t}\right)^2\right] \quad \text{for} \quad t > 0. \tag{4.17}$$

This function q approaches zero as $t \to \infty$, but (4.10) fails to hold, and the resulting equation (4.9) has the unbounded solution u_3 given as

$$u_3(t) = \sqrt{t}\cdot\cos t \cdot \left[\exp \frac{1}{2}\int_1^t \frac{\cos 2s}{s}\,ds\right] \quad \text{for } t > 0. \tag{4.18}$$

EXAMPLE 4.2. Consider the *Bessel equation*

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left[1 - \frac{\nu^2}{t^2}\right] y = 0 \tag{4.19}$$

for $t > 0$, where ν is a given parameter. The Sturm transformation

$$u(t) = \sqrt{t}\, y(t) \tag{4.20}$$

can be used to rewrite (4.19) in terms of u as

$$\frac{d^2u}{dt^2} + \left[1 + \frac{(1/4) - \nu^2}{t^2}\right] u = 0, \tag{4.21}$$

where this latter equation is of the oscillatory type (1.6) with

$$p(t) := 1 \qquad \text{and} \qquad q(t) := \frac{\nu^2 - (1/4)}{t^2}. \tag{4.22}$$

Hence (4.21), (4.22) is a special case of the previous equation (4.9), (4.10), and so the results of Example 4.1 can be applied here with

$$\mu(t) = |\nu^2 - \tfrac{1}{4}| \cdot t^{-1}.$$

One finds the solutions

$$
\begin{aligned}
u_1(t) &= e^{-i\left[t + \frac{(1/4)-\nu^2}{2t}\right]}[1 + E_1(t)],\\
u_1'(t) &= -ie^{-i\left[t + \frac{(1/4)-\nu^2}{2t}\right]}[1 + F_1(t)],
\end{aligned} \tag{4.23}
$$

and

$$u_2(t) = e^{+i\left[t + \frac{(1/4) - \nu^2}{2t}\right]}[1 + E_2(t)],$$
$$u_2'(t) = +ie^{+i\left[t + \frac{(1/4) - \nu^2}{2t}\right]}[1 + F_2(t)] \tag{4.24}$$

for $t \geqslant |\nu^2 - \frac{1}{4}|$, with the error bounds

$$|E_1(t)|, |F_1(t)| \leqslant \frac{|\nu^2 - (1/4)|}{t}\left[1 + \frac{|\nu^2 - (1/4)|}{4t}\right]\exp(\tfrac{1}{4}|\nu^2 - \tfrac{1}{4}|^2 \cdot t^{-2}) \tag{4.25}$$

and

$$|E_2(t)|, |F_2(t)|$$
$$\leqslant \frac{|\nu^2 - (1/4)|}{2t}\left[\exp\left(\left|\nu^2 - \frac{1}{4}\right| \cdot t^{-1}\right) + \frac{|\nu^2 - (1/4)|}{2t}\exp\left(\frac{1}{4}\left|\nu^2 - \frac{1}{4}\right|^2 \cdot t^{-2}\right)\right] \tag{4.26}$$

for $t \geqslant |\nu^2 - \frac{1}{4}|$. In particular one has explicit versions of the order estimates

$$E_j(t), F_j(t) = O\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty \quad (j = 1, 2). \tag{4.27}$$

These results can be used to provide information on the asymptotic behavior of the Bessel functions for $t \geqslant |\nu^2 - \frac{1}{4}|$. Moreover, the approach can be repeated on a contiguous interval $[c, |v^2 - \frac{1}{4}|)$ for a suitable positive $c < |\nu^2 - \frac{1}{4}|$, and in this way the approach can be used to study the Bessel functions for $t > 0$.

In particular, from (4.20) one has fundamental solutions y_1, y_2 for the Bessel equation (4.19), given as

$$y_1(t) = t^{-1/2}u_1(t) \qquad \text{and} \qquad y_2(t) = t^{-1/2}u_2(t), \tag{4.28}$$

where u_1 and u_2 are given by (4.23)–(4.27). Alternatively, taking the real and imaginary parts, one has suitable real-valued fundamental solutions y_3 and y_4 satisfying estimates which imply in particular the crude resuls

$$y_3(t) = (1/\sqrt{t})[\cos t + O(\tfrac{1}{t})] \text{ and } y_4(t) = (1/\sqrt{t})[\sin t + O(\tfrac{1}{t})] \tag{4.29}$$

as $t \to \infty$. Note that the error terms are identically zero if $\nu = \pm\frac{1}{2}$ (cf. (4.25), 4.26)).

Note added in proof. The Riccati transformation can also be used to obtain Liouville–Green approximations for related vector systems, such as the system

$$\frac{d^2u}{dt^2} = P(t)u \tag{4.30}$$

for a vector-valued solution function $u = u(t)$ and for suitable $n \times n$ matrix-valued coefficient functions $P = P(t)$. Such results are the subject of a separate study to appear elsewhere.

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